arXiv:0912.2113v3 [math-ph] 9 Apr 2010

ON SCHRÖDINGER EQUATION WITH TIME-DEPENDENT QUADRATIC HAMILTONIAN IN R^d

ERWIN SUAZO

Abstract. We study solutions to the Cauchy problem for the linear and nonlinear Schrödinger equation with a quadratic Hamiltonian depending on time. For the linear case the evolution operator can be expressed as an integral operator with the explicit formula for the kernel. As a consequence, conditions for local and global in time Strichartz estimates can be established. For the nonlinear case we show local well-posedness. As a particular case we obtain well-posedness for the damped harmonic nonlinear Schrödinger equation.

1. INTRODUCTION

In this paper we first study the time-dependent linear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\left(t\right)\psi, \qquad \psi(x,0)=\varphi \tag{1.1}$$

with the quadratic Hamiltonian

$$H(t)\psi = -\frac{1}{2}\Delta\psi + \sum_{j=1}^{d}\left(\frac{b_j\left(t\right)}{2}x_j^2\psi - f_j(t)x_j\psi + ig_j(t)\frac{\partial \psi}{\partial x_j} - i\frac{c_j\left(t\right)}{2}\left(2x_j\frac{\partial \psi}{\partial x_j}+\psi\right)\right), \tag{1.2}$$

where $b_j, f_j, g_j, c_j \in C^1$ (b_j, f_j, g_j could be piecewise continuous functions) and $\varphi \in S(\mathbb{R}^d)$ ($S(\mathbb{R}^d)$ is the Schwartz space) to simplify the discussion. We derive its evolution operator given by an explicit formula in the form

$$\psi\left(x,t\right) = U_H(t,0)\varphi\left(x\right) = \int_{\mathbb{R}^d} G_H(x,y,t)\ \varphi\left(y\right)\ dy \tag{1.3}$$

by constructing the fundamental solution (FS) associated to (1.1)-(1.2), see Lemma 1; the FS is a solution of (1.1) with the initial data $G_H(x,y,0)=\delta(x-y)$. Next we study properties of U_H and give conditions to obtain local and global in time Strichartz estimates for $\varphi \in L^2_x(\mathbb{R}^d)$. We prove that the nonlinear version of (1.1)-(1.2) with algebraic nonlinearity is locally wellposed in $L^2_x(\mathbb{R}^d)$ in the subcritical sense, see Section 4. Finally, we introduce the damped harmonic nonlinear Schrödinger equation:

$$i\frac{\partial u}{\partial t} = \frac{\omega_0}{2}\left(-\frac{\partial^2 u}{\partial x^2}+x^2u\right) + i\frac{\lambda}{2}\left(2x\frac{\partial u}{\partial x}+u\right) + h|u|^{p-1}u \tag{1.4}$$

and also prove well-posedness. A systematic study of the blow up and scattering results of (1.1)-(1.2) are presented in [52].

1991 *Mathematics Subject Classification.* Primary 35A08, 35Q40, 35Q55. Secondary 81Q05.

Key words and phrases. The Cauchy initial value problem, Schrödinger equation (mathematical properties), Quadratic Hamiltonian, Fundamental solution, Propagator, Green function, Mehler's formula, Strichartz estimates, Nonlinear Schrödinger equation.

The study of time dependent quadratic Hamiltonians is quite complex and has brought a great deal of attention see [24], [55] and references therein, see also [23] for a nice discussion on this type of Hamiltonians consider "folk wisdow". The study of methods (i.e. propagator method, dynamical invariant and second quatization methods) used to find the exact propagators for Schrödinger equations has also been studied by several authors [55]. The equation (1.1) can be solved, at least formally, using a time evolution operator, $U(t, t_0)$ given by

$$U(t,t_0) = \sum_{k=0}^{n} (-i)^k \int_{t_0}^{t_1} dt_1 ... \int_{t_0}^{t_{k-1}} dt_k H(t_1)...H(t_k) = \mathrm{T}\left(\exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} H(t')\ dt'\right)\right), \tag{1.5}$$

where T is the time ordering operator which orders operators with larger times to the left, and of course this expression might diverge.

The fundamental solution G_H for the equation (1.1)-(1.2) includes the following examples with the explicit expressions:

Table I. Some exactly solvable quadratic Hamiltonians (We assume E, k constants).

Hamiltonian $H(t)$	Fundamental Solution (Propagator)
Free Particle $H_0(t)\psi = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}$	$G_0(x,y,t) = \frac{1}{\sqrt{2\pi i \sin t}} e^{i\|x-y\|^2/2t}$
Constant Electric Field $H_1(t)\psi = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + E\cdot x\psi$	$G_1(x,y,t) = \frac{1}{\sqrt{2\pi i \sin t}} \exp\left(\frac{i(x-y)^2}{2t}\right)$ $\times \exp\left(\frac{iE(x+y)}{2}t - \frac{iE^2}{24}t^3\right)$
Isotropic Oscillator $H_2(t)\psi = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}x^2\psi$	$G_2(x,y,t) = \frac{1}{\sqrt{2\pi i \sin t}}$ $\times \exp\left(i\frac{1}{4\sin(t)}\left((x^2+y^2)\cos t - 2xy\right)\right)$
Repulsive harmonic potential $H_3(t)\psi = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} - \frac{1}{2}x^2\psi$	$G_3(x,y,t) = \frac{1}{\sqrt{2\pi i \sinh t}}$ $\times \exp\left(i\frac{1}{4\sinh t}\left((x^2+y^2)\cosh t - 2xy\right)\right)$
Anisotropic Oscillator $H_4(t)\psi = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}\omega^2 x^2\psi$	$G_4(x,y,t) = \frac{\omega}{\sqrt{2\pi i \sin \omega t}}$ $\times \exp\left(i\frac{\omega}{4\sin(\omega t)}\left((x^2+y^2)\cos\omega t - 2xy\right)\right)$
Modified MC-SS Oscillator $H_6(t)\psi = -\cos^2 t\frac{\partial^2\psi}{\partial x^2} + \sin^2 t x^2\psi$ $-i\frac{\sin 2t}{2}\left(2x\frac{\partial}{\partial x} - 1\right)\psi$	$G_6(x,y,t) = \frac{1}{\sqrt{2\pi i(\cos t\sinh t + \sin t\cosh t)}}$ $\times \exp\left(\frac{\left(x^2-y^2\right)\sin t\sinh t + 2xy - \left(x^2+y^2\right)\cos t\cosh t}{2i(\cos t\sinh t+\sin t\cosh t)}\right)$
Damped Harmonic Oscillator $H_7(t)\psi = -\frac{\omega_0}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{\omega_0}{2}x^2\psi$ $+i\frac{\lambda}{2}\left(2x\frac{\partial}{\partial x} + 1\right)\psi$	$G_7(x,y,t) = \sqrt{\frac{\omega}{2\pi i\omega_0 \sin\omega t}}$ $\times \exp\left(\frac{i\omega}{2\omega_0\sin\omega t}\left((x^2+y^2)\cos\omega t - 2xy\right)\right)$ $\times \exp\left(\frac{i\lambda}{2\omega_0}(x^2-y^2)\right),\ \omega = \sqrt{\omega_0^2-\lambda^2} > 0$
Analog of Heat Equation with Linear Drift $H_8(t)\psi = -\frac{\partial^2\psi}{\partial x^2} - ikx\frac{\partial\psi}{\partial x}, \quad k > 0$	$G_8(x,y,t) = \frac{\sqrt{k}e^{kt/2}}{\sqrt{2\pi i \sinh(kt)}} \exp\left(\frac{ike^{kt}\left[e^{-kt}x - e^{kt}y\right]^2}{4\sinh(kt)}\right)$

The generality of equation (1.1)-(1.2) includes several examples of interest not only for linear [18], [55], but for nonlinear Schrödinger equations (NLS) [5], [6]; for a general review in NLS see [8], [56]. The explicit solution formula for (1.1)-(1.2), see (2.10), allows us to treat nonlinear versions of these cases with the advantage of allowing time dependence factors so that questions such as local and global existence, finite time blow up and scattering can be solved using similar methods to those

used in the case of the nonlinear Schrödinger equation without potential. The results presented here are of interest to study the behavior in time of singularities of solutions for Schrödinger equations in the same direction as in [61]. Also the fact that in quantum mechanics it is rare to find an exact solution to nonstationary problems, see [43], [42], makes our explicit solution useful for testing numerical methods to solve time-dependent Schrödinger equation. One of the original motivations of this paper is to introduce a generalization of the practical formula (1.19)-(1.20), for applications see [5], [61] and [33].

1.1. **Quadratic Quantum Hamiltonians.** The expert would recognize (1.2) as a quantum mechanical self-adjoint Hamiltonian, which is a quadratic polynomial in x and $p = -i\partial/\partial x$ with time dependent coefficients [23], [24]:

$$H(t) = a(t)p^2 + \frac{c(t)}{2}(p \cdot x + x \cdot p) + \frac{b(t)}{2}x^2 - g(t)p - f(t)x + \zeta(t). \tag{1.6}$$

As pointed out in [23] one can assume $\zeta(t) = 0$ since it causes a trivial phase factor in the propagator. We also have assumed in (1.1) $a(t) = 1/2$, thinking we can disregard it after a substitution, however there are important cases that we will be missing, for example the Caldirola-Kanai Hamiltonian [4], [29]. In [12] the authors did not know about this case and called it "third model," it was one the models of the damped harmonic oscillator with explicit propagator considered in that publication. The Caldirola-Kanai Hamiltonian was introduced more than 60 years ago [4], [29]:

$$H_{CK}(t)\psi = -\frac{\omega_0 e^{-\lambda t}}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{\omega_0 e^{\lambda t}}{2}x^2\psi. \qquad \text{(Caldirola-Kanai Hamiltonian)}$$

its fundamental solution is given by

$$G_{CK}(x,y,t) = \sqrt{\frac{\omega e^{\lambda t}}{2\pi i \omega_0 \sin \omega t}}\, e^{i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2)}, \tag{1.7}$$

where

$$\alpha(t) = \frac{\omega \cos \omega t - \lambda \sin \omega t}{2\omega_0 \sin \omega t} e^{2\lambda t}, \tag{1.8}$$

$$\beta(t) = -\frac{\omega}{\omega_0 \sin \omega t} e^{\lambda t}, \tag{1.9}$$

$$\gamma(t) = \frac{\omega \cos \omega t + \lambda \sin \omega t}{2\omega_0 \sin \omega t}. \tag{1.10}$$

This model has been studied for several authors, and a detailed bibliography can be found in [55].

Another example with $a(t)$ not constant that can be solved using the same ideas presented in this work is

$$H_{10}(t)\psi = -a(t)\frac{\partial^2 \psi}{\partial x^2} + \frac{b(t)}{2}x^2\psi, \tag{1.11}$$

$$a(t) = \frac{(\Omega^2 \cos(\Omega t) - \gamma \sin(\Omega t)\tanh(\gamma t))}{\cosh(\gamma t)(\cos(\gamma t)\cosh(\gamma t) - 2\gamma)}, \tag{1.12}$$

$$b(t) = -\frac{\omega^2}{4a(t)}, \ \Omega = \sqrt{\omega^2 - \gamma^2}, \tag{1.13}$$

and its fundamental solution is given by

$$G_{10}(x,y,t)=\sqrt{\frac{m_0\Omega\cosh\gamma t}{2\pi i\sin(\Omega t)}}e^{i(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2)}, \tag{1.14}$$

with

$$\alpha(t)=\frac{\cosh(\gamma t)\left(m_0\Omega\cosh(\gamma t)\cos(\Omega t)-\gamma\right)}{2\sin(\Omega t)}, \tag{1.15}$$

$$\beta(t)=-\frac{m_0\Omega\cosh\gamma t}{2\pi i\sin(\Omega t)}, \tag{1.16}$$

$$\gamma(t)=\frac{m_0\Omega\cos(\Omega t)}{2\sin(\Omega t)}. \tag{1.17}$$

Suslov et. al in [13] study the quantum integrals of motion between others of these last two models.

1.2. **A General Formula.** If we consider the linear Schrödinger equation on $\mathbb{R}^n$ (1.1)-(1.2) with

$$H_V(t)\psi=-\frac{1}{2}\Delta\psi+V(x)\psi, \tag{1.18}$$

where $V(x)=\sum_{j=1}^n\left(\delta_j\frac{\omega_j^2}{2}x_j^2+b_jx_j\right)$, $n\geq 1$, $\omega_j>0$, $\delta_j\in\{-1,0,1\}$, then the solution is given by the following formula [5]

$$\psi(x,t)=U_V(t)f:=\frac{1}{\sqrt{2i\pi g_j(t)}}\int_{\mathbb{R}^n}e^{iS_V(x,y,t)}f(y)dy \tag{1.19}$$

where

$$S_V(x,y,t)=\frac{1}{g_j(t)}\left(\frac{x_j^2+y_j^2}{2}h_j(t)-x_jy_j\right)$$

$$\{g_j(t),h_j(t)\}=\begin{cases}\left\{\frac{\sinh(\omega_jt)}{\omega_j},\cosh(\omega_jt)\right\}, & \text{if } \delta_j=-1,\\ \{t,1\}, & \text{if } \delta_j=0,\\ \left\{\frac{\sin(\omega_jt)}{\omega_j},\cos(\omega_jt)\right\}, & \text{if } \delta_j=+1.\end{cases} \tag{1.20}$$

This formula includes the free particle propagator introduced by Ehrenfest [17] that corresponds to the case $\delta_j=0$ for all j in (1.18). The propagator for the simple harmonic oscillator is obtained from (1.19), by choosing $\delta_j=1$ for all j in (1.18) (a consequence of Mehler's formula for Hermite polynomials [18]). Finally, the propagator for the isotropic harmonic potential is obtained by choosing $\delta_j=-1$ for all j in (1.18).

The fact that we want to emphasize in formulas (1.19)-(1.20) is that it combines the fundamental solutions of the free particle and simple and isotropic harmonic oscillators in the one dimensional case (using tensor product) to construct the explicit solutions from a variety of Schrödinger equations in several dimensions. One motivation comes from the Schrödinger equation (1.1)-(1.2) with the Hamiltonian

$$H_h(t)\psi=-\frac{1}{2}\Delta\psi+1/2\left(-\omega_1x_1^2+\omega_2x_2^2\right)\psi, \tag{1.21}$$

(compare with H_3 and H_4 from Table I) whose evolution operator satisfies global in time Strichartz estimates, see [5].

However, the explicit formula for the evolution operator corresponding to (1.1)-(1.2) allows us to consider a wider range of operators to solve explicitly a variety of Schrödinger equations in several dimensions.

1.3. **Cases from Table I Not Included in Formula (1.19)-(1.20).** The case of the particle in a constant external field having the Hamiltonian $H_1(t)$ case was studied in detail by [15], [1], [3], [18], [25], [41] and [45]. The forced harmonic oscillator is obtained in (1.1)-(1.2) by choosing

$$H(t)\psi = \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial x^2}+x^2\right)\psi + \frac{\hbar}{\sqrt{2}}\left(\delta(t)\left(x-\frac{\partial}{\partial x}\right)+\delta^*(t)\left(x+\frac{\partial}{\partial x}\right)\right)\psi,$$

where $\delta(t)$ is a complex valued function of time t and the symbol $*$ denotes complex conjugation. It corresponds to the case which is of interest in many advanced problems, examples including polyatomic molecules in varying external fields, crystals through which an electron is passing and exciting the oscillator modes, and other interactions of the modes with external fields. An example of this type of Hamiltonian is $H_6(t)$, introduced in [10].

The damped oscillations have been analyzed to a great extent in classical mechanics; see, for example, [2] and [34]. In [12] the damped harmonic Schrödinger equation with self-adjoint Hamiltonian $H_7(t)$ was considered. The case $H_8(t)$ is taken from the analogous case of the heat equation with linear drift, see [38] for a proof of this fundamental solution using Lie theory approach.

All examples in Table I are explicit solutions because one can solve the characteristic equation (2.2) and evaluate the integrals (2.5)-(2.8) explicitly. Solving (2.2) could require elaborated techniques, see for example [35]. However, even without solving (2.2) explicitly important conclusions can be drawn. For example, see [7], where

$$H_{\tilde{V}}(t)\psi = -\frac{1}{2}\Delta\psi + \tilde{V}(x,t)\psi \tag{1.22}$$

with $\tilde{V}(x,t)\psi = \sum_{j=1}^{d}\frac{b_j(t)}{2}x_j^2\psi$ was considered and finite time blow up results are obtained. Another interesting example is the parametric forced harmonic oscillator with a time-dependent frequency $\omega(t)$, see [42]

$$H(t)\psi = -\frac{1}{2}\triangle\psi + \frac{m}{2}\omega(t)^2x^2\psi - f(t)x\psi. \tag{1.23}$$

This case is relevant in the context of charged particle traps [21].

1.4. **Organization of the paper.** In Section 2 we derive the explicit formula for the fundamental solution of (1.1) with Hamiltonian (1.2) (Lemma 1). We also present explicitly the time evolution operator and relevant properties (Corollary 1), conditions on the uniqueness of the solution for (1.1)-(1.2), and continuous dependence on the initial data and smoothness of the solution (Theorem 1). In Section 3 we discuss Strichartz type estimates for U_H (Lemma 5 and Theorem 2). Finally, in Section 4 we discuss local well-posedness in $L^2_x(\mathbb{R}^d)$ for the nonlinear case in the subcritical sense (Proposition 1).

2. LINEAR CASE

In this section we wish to prove a generalization of the formula (1.19)-(1.20) in $\mathbb{R}^d$:

Lemma 1. *(Fundamental Solution) 1. Let $\varphi \in S(\mathbb{R}^d)$. The Cauchy initial value problem (1.1)-(1.2) has the following fundamental solution*

$$G_H(x,y,t) = \left(\prod_{j=2}^{d} \frac{1}{2\pi i \mu_j(t)}\right)^{d/2} e^{i\left(\sum \alpha_j(t)x_j^2+\beta_j(t)x_jy_j+\gamma_j(t)y_j^2+\delta_j(t)x_j+\varepsilon_j(t)y_j+\kappa_j(t)\right)}. \tag{2.1}$$

μ_j satisfies

$$\mu_j'' + 4\sigma_j(t)\mu_j = 0, \tag{2.2}$$

with $\sigma_j(t) = b_j(t)/2 - c_j^2(t)/4 - c_j'(t)/4$, which must be solved subject to $\mu_j(0) = 0$, $\mu_j'(0) = 1$. Furthermore, $\alpha_j(t), \beta_j(t), \gamma_j(t), \delta_j(t), \varepsilon_j(t), \kappa_j(t)$ are differentiable in time t only and are given explicitly by

$$\alpha_j(t) = \frac{1}{2}\frac{\mu_j'(t)}{\mu_j(t)} - \frac{c_j(t)}{2}, \tag{2.3}$$

$$\beta_j(t) = -\frac{1}{\mu_j(t)}, \tag{2.4}$$

$$\gamma_j(t) = \frac{1}{2\mu_j(t)\mu_j'(t)} - 2\int_0^t \frac{\sigma_j(\tau)}{\left(\mu_j'(\tau)\right)^2}\, d\tau + \frac{c_j(0)}{2} \tag{2.5}$$

$$\delta_j(t) = \frac{1}{\mu_j(t)}\int_0^t \left(f_j(\tau) - c_j(\tau)g_j(\tau)\right)\mu_j(\tau) + g_j(\tau)\mu_j'(\tau)\ \, d\tau, \tag{2.6}$$

$$\begin{aligned} \varepsilon_j(t) &= -\frac{\delta_j(t)}{\mu_j'(t)} + 4\int_0^t \frac{\mu_j(\tau)\delta_j(\tau)\sigma_j(\tau)}{\left(\mu_j'(\tau)\right)^2} d\tau \\ &\quad + \int_0^t \frac{1}{\mu'(\tau)}\left(f_j(\tau) - c_j(\tau)g_j(\tau)\right)\, d\tau, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \kappa_j(t) &= \frac{\mu_j(t)}{2\mu_j'(t)}\delta_j^2(t) - 2\int_0^t \frac{\sigma_j(\tau)}{\left(\mu_j'(\tau)\right)^2}\left(\mu_j(\tau)\delta_j(\tau)\right)^2\, d\tau \\ &\quad - \int_0^t \frac{\mu_j(\tau)\delta_j(\tau)}{\mu_j'(\tau)}\left(f_j(\tau) - c_j(\tau)g_j(\tau)\right)\, d\tau \end{aligned} \tag{2.8}$$

with

$$\delta_j(0) = g_j(0), \qquad \varepsilon_j(0) = -\delta_j(0), \qquad \kappa_j(0) = 0. \tag{2.9}$$

2. Convergence to the initial data:

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} G_H(x,y,t)\psi(y,t)dy = \psi(x,0).$$

Thus, the fundamental solution (propagator) is explicitly given by (2.14) in terms of the characteristic function (2.2) with (2.3)-(2.8).

Remark 1. *The conditions (2.9), which are justified since we are looking for the following asymptotic formula, see* [52], *holds:*

$$\frac{e^{i(\alpha_j(t)x_j^2+\beta_j(t)x_jy_j+\gamma_j(t)y_j^2+\delta_j(t)x_j+\varepsilon_j(t)y_j+\kappa_j(t))}}{\sqrt{2\pi i\mu_j(t)}} \to \frac{1}{\sqrt{2\pi i t}}\exp\left(i\frac{(x_j-y_j)^2}{2t}\right)\exp\left(ig_j(0)(x_j-y_j)\right)$$

$$\times \exp\left(-\frac{ic_j(0)}{2}(x_j^2 - y_j^2)\right), \; t \to 0^+.$$

Theorem 1. *1. Let $\varphi \in S(\mathbb{R}^d)$, then the Cauchy initial value problem (1.1)-(1.2) has the following unitary evolution operator:*

$$U_H(t)\varphi \equiv \left(\prod_{j=2}^{d} \frac{1}{2\pi i \mu_j(t)}\right)^{d/2} \int_{\mathbb{R}^d} e^{i\left(\sum \alpha_j(t)x_j^2 + \beta_j(t)x_j y_j + \gamma_j(t) y_j^2 + \delta_j(t)x_j + \varepsilon_j(t) y_j + \kappa_j(t)\right)} \varphi(y)dy. \tag{2.10}$$

2. If $\varphi \in S(\mathbb{R}^d)$, then $U_H(t)\varphi \in S(\mathbb{R}^d)$.

3. $U_H(t,s) = U_H(t)U_H^{-1}(s)$ and by duality $U_H(t,s)$ can be extended to $S'(\mathbb{R}^d)$. Furthermore, $U_H(\cdot)\varphi \in C(\mathbb{R}, S'(\mathbb{R}^d))$ for every $\varphi \in S'(\mathbb{R}^d)$.

If ψ satisfies (1.1)-(1.2) and it is smooth, then:

4. The following estimates hold:

$$\|U_H(t)\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}, \tag{2.11}$$

$$\|U_H(t,s)\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \left(\prod_{j=1}^{d} \frac{1}{\sqrt{4\pi i \mu_j(t)\mu_j(s)(\gamma_j(s) - \gamma_j(t))}}\right)^{d/2} \|\varphi\|_{L^1(\mathbb{R}^d)}. \tag{2.12}$$

5. Uniqueness and continuous dependence on the initial data in $L_x^2(\mathbb{R}^d)$ holds.

Remark 2. *The function μ_j will characterize the singularities* [14], [30], [61] *and* [58]. *More complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions. For a nice conection between the characteristic equation (2.2) and Ehrenfest Theorems see* [13].

Corollary 1. *The evolution operator associated to (1.1)-(1.2) satisfies the following properties:*

1. *$U_H(t,s) = U_H(t)U_H^{-1}(s)$.*
2. *$U_H(t,t) = Id$.*
3. *The map $(t,s) \to U_H(t,s)$ is strongly continuous.*
4. *$U_H(t,\tau)U_H(\tau,s) = U_H(t,s)$.*

2.1. **Proof of the Lemma 1.** We follow [11] where the fundamental solution is constructed for a more general case in $d=1$ dimension. The Lemma follows from the construction in the $1d$ case and then using tensor product to construct the d-dimensional fundamental solution. We recall how to construct the fundamental solution in the one dimensional case for

$$\frac{\partial \psi_j}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi_j}{\partial x_j^2} + \frac{b_j(t)}{2}x_j^2\psi_j - f_j(t)x_j\psi_j + ig_j(t)\frac{\partial \psi_j}{\partial x_j} - i\frac{c_j(t)}{2}\left(2x_j\frac{\partial \psi_j}{\partial x_j} - \psi_j\right). \tag{2.13}$$

The fundamental solution is found by using the ansatz

$$\psi_j = A_j e^{iS_j} = A_j(t)\, e^{iS_j(x,y,t)} \tag{2.14}$$

with

$$A_j = A_j(t) = \frac{1}{\sqrt{2\pi i \mu_j(t)}} \tag{2.15}$$

and

$$S_j = S_j\left(x,y,t\right) = \alpha_j\left(t\right)x_j^2 + \beta_j\left(t\right)x_jy_j + \gamma_j\left(t\right)y_j^2 + \delta_j\left(t\right)x_j + \varepsilon_j\left(t\right)y_j + \kappa_j\left(t\right), \tag{2.16}$$

where $\alpha_j\left(t\right), \beta_j\left(t\right), \gamma_j\left(t\right), \delta_j\left(t\right), \varepsilon_j\left(t\right)$, and $\kappa_j\left(t\right)$ are differentiable real-valued functions of time t only. Indeed,

$$\frac{\partial S_j}{\partial t} = -\frac{1}{2}\left(\frac{\partial S_j}{\partial x}\right)^2 - b_jx_j^2 + f_jx_j + \left(g_j - c_jx_j\right)\frac{\partial S_j}{\partial x_j} \tag{2.17}$$

by choosing

$$\frac{\mu_j'}{2\mu_j} = \frac{1}{2}\frac{\partial^2 S_j}{\partial x_j^2} + \frac{c_j}{2} = \alpha_j\left(t\right) + \frac{c_j(t)}{2}. \tag{2.18}$$

Equating the coefficients of all admissible powers of $x_j^m y_j^n$ with $0 \leq m+n \leq 2$ gives the following system of ordinary differential equations:

$$\frac{d\alpha_j}{dt} + b_j\left(t\right) + 2c_j\left(t\right)\alpha_j + 2\alpha_j^2 = 0, \tag{2.19}$$

$$\frac{d\beta_j}{dt} + \left(c_j\left(t\right) + 2\alpha_j\left(t\right)\right)\beta_j = 0, \tag{2.20}$$

$$\frac{d\gamma_j}{dt} + \frac{\beta_j^2\left(t\right)}{2} = 0, \tag{2.21}$$

$$\frac{d\delta_j}{dt} + \left(c_j\left(t\right) + 2\alpha_j\left(t\right)\right)\delta_j = f_j\left(t\right) + 2\alpha_j\left(t\right)g_j\left(t\right), \tag{2.22}$$

$$\frac{d\varepsilon_j}{dt} = \left(g_j\left(t\right) - \delta_j\left(t\right)\right)\beta_j\left(t\right), \tag{2.23}$$

$$\frac{d\kappa_j}{dt} = g_j\left(t\right)\delta_j\left(t\right) - \frac{\delta_j^2\left(t\right)}{2}, \tag{2.24}$$

where the first equation is the Riccati nonlinear differential equation. Substituting (2.18) into (2.19) results in the second order linear equation

$$\mu_j'' + 4\sigma_j\left(t\right)\mu_j = 0 \tag{2.25}$$

with

$$\sigma_j\left(t\right) = \frac{b_j(t)}{2} - \frac{c_j^2(t)}{4} - \frac{c_j'}{4}, \tag{2.26}$$

which must be solved subject to the initial data

$$\mu_j\left(0\right) = 0, \qquad \mu_j'\left(0\right) = 1. \tag{2.27}$$

We shall refer to equation (2.25) as the *characteristic equation* and its solution $\mu_j\left(t\right)$, subject to (2.27), as the *characteristic function.* Using integration by parts we can solve (2.19)-(2.24) obtaining (2.3)-(2.8).

Part 2 is a consequence of the two following results in the one dimensional case proven in [52].

Lemma 2. *Let G be defined by (2.1). There exists a complex-valued function K satisfying*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,z,t)K(z,y,0)\chi(y)dzdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,z,t)K(z,y,0)\chi(y)dydz \tag{2.28}$$

and

$$\int_{\mathbb{R}^d} G(x,y,t)\psi(y,t)dy = \int_{\mathbb{R}^d} K(x,y,t)\chi(y)dy. \tag{2.29}$$

Lemma 3. *If we consider the intial data $\psi(x,0)$ such that $\psi(x,0) = \int_{\mathbb{R}^d} K(x,y,0)\chi(y)dy$ for some $\chi \in L^1(\mathbb{R}^d)$, then*

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} G(x,y,t)\psi(y,t)dy = \psi(x,0).$$

Remark 3. *For the case of the free particle propagator $K(x,y,0) = e^{ix\cdot y}$ and $\chi(y)$ is the Fourier transform of $\psi(x,0)$.*

2.2. **Proof of Theorem.** 1. The propagator $U_H(t)$ for the equation (1.1)-(1.2) can be written as

$$U_H(t)\varphi = A_t B_t \mathfrak{F}\left(C_t\varphi\right), \tag{2.30}$$

where $A_t(x) = e^{i(\alpha(t)x^2+\delta(t)x+\kappa(t))}$, $C_t(x) = e^{i(\gamma(t)x^2+\varepsilon(t)x)}$, $B_t w(x) = (2\pi i\mu(t))^{-\frac{1}{2}}\, w\left(-\beta(t)x/2\pi\right)$ and $\mathfrak{F}$ is the Fourier transform. Since the Fourier transform is an isomorphism on the Schwartz space we have the operator $U_H(t)$ is an isomorphism on the Schwartz space.

2. We claim that

$$\psi\left(x,0\right) = U_H^{-1}\left(t\right)\psi\left(x,t\right) = \int_{-\infty}^{\infty} H\left(x,y,t\right)\ \psi\left(y,t\right)\ dy, \tag{2.31}$$

where

$$H\left(x,y,t\right) = \left(\prod_{j=1}^{d} \frac{1}{-2\pi i\mu_j\left(t\right)}\right)^{d/2} e^{-i\sum_{j=1}^{d} S_j(y_j,x_j,t)} \tag{2.32}$$

such that

$$U_H\left(t\right)U_H^{-1}\left(t\right) = U_H^{-1}\left(t\right)U_H\left(t\right) = I = \mathrm{id}. \tag{2.33}$$

First we observe that the following orthogonality relations of the kernels hold:

$$\int_{\mathbb{R}^d} G\left(x,y,t\right)H\left(y,z,t\right)\ dy \ = \ e^{i\sum_{j=1}^{d}(\alpha_j(t)(x_j+z_j)+\delta_j(t))(x_j-z_j)} \prod_{j=1}^{d} \delta\left(x_j - z_j\right), \tag{2.34}$$

$$\int_{\mathbb{R}^d} H\left(x,y,t\right)G\left(y,z,t\right)\ dy \ = \ e^{-i\sum_{j=1}^{d}(\gamma_j(t)(x_j+z_j)+\varepsilon_j(t))(x_j-z_j)} \prod_{j=1}^{d} \delta\left(x_j - z_j\right), \tag{2.35}$$

where $\delta\left(x\right)$ is the Dirac delta function with respect to the space coordinates.

Next, we have

$$\begin{aligned}
&U_H^{-1}\left(t\right)U_H\left(t\right)\psi\left(x,0\right) = U_H^{-1}\left(t\right)\psi\left(x,t\right)\\
&\quad= \int_{\mathbb{R}^d} H\left(x,y,t\right)\ \psi\left(y,t\right)\ dy\\
&\quad= \int_{\mathbb{R}^d} H\left(x,y,t\right)\ \left(\int_{\mathbb{R}^d} G_H\left(y,z,t\right)\ \psi\left(z,0\right)\ dz\right)\ dy\\
&\quad= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} H\left(x,y,t\right)G_H\left(y,z,t\right)\ dy\right)\ \psi\left(z,0\right)\ dz
\end{aligned}$$

$$= \int_{\mathbb{R}^d} e^{-i\sum_{j=1}^{d}(\gamma_j(t)(x_j+z_j)+\varepsilon_j(t))(x_j-z_j)} \prod_{j=1}^{d} \delta\left(x_j - z_j\right)\ \psi\left(z,0\right)\ dz$$
$$= \psi\left(x,0\right),$$

or $U_H^{-1}(t)\, U_H(t) = I$. A formal proof of the second relation $U_H(t)\, U_H^{-1}(t) = I$ is similar. The rest of the statement follows for a standard duality argument, see [8].

3. This claim is a consequence of (1) multiplying the equation

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - f(t)x\psi - i(c(t)x - g(t))\partial_x\psi - i\frac{c(t)}{2}\psi$$

by $\overline{\psi}$ (2) integrating in the space variable, and (3) equating the imaginary parts of both sides to obtain

$$Re\int \psi_t\overline{\psi}dx = -\frac{1}{2}\int\left(c(t)x - g(t)\right)\partial_x(|\psi|^2)dx - \frac{1}{2}\int c(t)\,|\psi|^2dx; \tag{2.36}$$

the estimate now follows from the solution being smooth.

We introduce the integral operator $U_H(t,s) = U_H(t)\, U_H^{-1}(s)$ by

$$U_H(t)\, U_H^{-1}(s)\, \psi(x,s) = \int_{-\infty}^{\infty} G(x,y,t,s)\, \psi(y,s)\ dy \tag{2.37}$$

with the kernel given by

$$G(x,y,t,s) = \int_{\mathbb{R}^d} G_H(x,z,t)\, H(z,y,s)\ dz. \tag{2.38}$$

Here,

$$G(x,y,t,s) = \left(\prod_{j=1}^{d} \frac{1}{\sqrt{4\pi i\mu_j(t)\,\mu_j(s)\left(\gamma_j(s) - \gamma_j(t)\right)}}\right)^{d/2} \tag{2.39}$$
$$\times \sum_{j=1}^{d} \exp\left(i\left(\alpha_j(t)\, x_j^2 - \alpha_j(s)\, y_j^2 + \delta_j(t)\, x_j - \delta_j(s)\, y_j + \kappa_j(t) - \kappa_j(s)\right)\right)$$
$$\times \sum_{j=1}^{d} \exp\left(\frac{\left(\beta_j(t)\, x_j - \beta_j(s)\, y_j + \varepsilon_j(t) - \varepsilon_j(s)\right)^2}{4i\left(\gamma_j(t) - \gamma_j(s)\right)}\right).$$

$$\begin{aligned}
\left|U_H(t,s)\,\psi(x,s)\right| &= \left|\int_{\mathbb{R}^d} G(x,y,t,s)\,\psi(y,s)\ dy\right| \\
&\leq \left(\prod_{j=1}^{d} \frac{1}{\sqrt{4\pi i\mu_j(t)\,\mu_j(s)\left(\gamma_j(s) - \gamma_j(t)\right)}}\right)^{d/2} \int_{\mathbb{R}^d} |\psi(y,s)|\ dy.
\end{aligned}$$

Thus, the estimate (2.12) holds.

4. The uniqueness of the solution and continuous dependence on the initial data follows by standard arguments using estimates of the type of (2.11).

3. EXAMPLES OF STRICHARTZ TYPE ESTIMATES

In this section we will discuss Strichartz type estimates for the operator U_H.

Definition 1. *We say that the exponent pair (q,r) is $\sigma-$admissible if $q,r \geq 2$, $(q,r,\sigma) \neq (2,\infty,1)$ and*

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}.$$

If equality holds we say that (q,r) is sharp σ-admissible, otherwise, we say that (q,r) is nonsharp σ-admissible. Note, in particular, that when $\sigma > 1$ the endpoint

$$P = \left(2, \frac{2\sigma}{\sigma-1}\right)$$

is sharp σ-admissible.

The following inequalities are known as Strichartz estimates studied by Strichartz, Ginibre, Velo, Keel and Tao and others, see [31] for the following version.

Lemma 4. *If $U(t) : H \to L^2(x)$, where H is a Hilbert space and*

- *For all $t > 0$ and $f \in H$, we have*
$$||U(t)f||_{L^2_x} \lesssim ||f||_H, \tag{3.1}$$
- *For some $\sigma > 0$, $t \neq s$ and $g \in L^1(x)$,*
$$||U(t)U(s)^*g||_\infty \lesssim |t-s|^{-\sigma}||g||_{L^1}. \tag{3.2}$$

Then

$$||U(t)f||_{L^q_tL^r_x} \lesssim ||f||_H \tag{3.3}$$
$$\left|\left|\int (U(s))^* F(s)ds\right|\right|_{L^2} \lesssim ||F||_{L^{q'}_tL^{r'}_x} \tag{3.4}$$
$$\left|\left|\int_{s<t} U(t)\,(U(s))^* F(s)ds\right|\right|_{L^q_tL^r_x} \lesssim ||F||_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x} \tag{3.5}$$

hold for all sharp σ-admissible exponent pairs (q,r), $(\tilde{q},\bar{r})$.

In order to obtain inequalities (3.3), (3.4), (3.5) for our operator U_H, we observe that the inequalities are valid if we replace (3.2) by

$$||U(t)\,(U(s))^* F(s)||_{L^\infty_x} \leq w(t-s)^\gamma||F(s)||_{L^1_x}, \qquad w \in L^1_\omega. \tag{3.6}$$

We can just mimic the proof of Lemma 3, see [56] and the adaptation in [5] done for the case U_V. For our case we deal with the inequality (3.6) and avoid the use of semigroup properties that we do not have. More specifically we obtain:

Lemma 5. *If $U_H(t)$ satisfies (3.6), then for any $T \in \bar{\mathbb{R}}_+$,*

$$||U_H(t)f||_{L^q_tL^r_x} \leq c_q||w{\cdot}\mathbf{1}_{(-2T,2T)}||^{\gamma_1}_{L^2}||f||_{L^2}, \tag{3.7}$$
$$\left|\left|\int (U_H(s))^* F(s)ds\right|\right|_{L^2} \lesssim \tilde{c}_{\tilde{q}}||w{\cdot}\mathbf{1}_{(-2T,2T)}||^{\gamma_2}_{L^2}||F||_{L^{q'}_tL^{r'}_x}, \tag{3.8}$$

$$\left|\left|\int_{s<t} U_H(t)\left(U_H(s)\right)^* F(s)ds\right|\right|_{L_t^q((-T,T);L_x^r)} \leq C_{q,\tilde{q}}||w\cdot\mathbf{1}_{(-2T,2T)}||_{L^2}^{\gamma_3}||F||_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}}, \tag{3.9}$$

holding for all sharp σ-admisible exponent pairs (q,r), $(\tilde{q},\bar{r})$, where γ_1, γ_2 , γ_3 are constants.

For the sake of clarity we outline the proof:

First, observe that (3.4) implies (3.3) by duality. Therefore we prove (3.4)

$$\left|\left|\int U_H(s)^*F(s)ds\right|\right|_{L^2(\mathbb{R}^2)} \lesssim ||F||_{L^{q'}(\mathbb{R},\ L^{r'}(\mathbb{R}^2))}.$$

By the TT* method this can be implied by the following inequality:

$$\left|\int\int < U_H(s)^*F(s), U_H(t)^*G(t)dsdt >\right| \lesssim ||F||_{L_t^{q'}L_x^{r'}}||G||_{L_t^{q'}L_x^{r'}}.$$

By symmetry, it suffices to prove

$$|T(F,G)| \leq ||F||_{L_t^{q'}L_x^{r'}}||G||_{L_t^{q'}L_x^{r'}} \tag{3.10}$$

where

$$|T(F,G)| = \int\int_{s<t} < U_H(s)^*F(s), U_H(t)^*G(t) > dsdt.$$

Since U_H is unitary ($U_H^* = U_H^{-1}$) on L^2, by Holder's inequality and the energy estimate, we get

$$\begin{aligned} |< U_H(s)^*F(s), U_H(t)^*G(t) >| &\leq ||U_H(s)^*F(s)||_{L_x^2}||U_H(t)^*G(t)||_{L_x^2} && (3.11)\\ &= ||F(s)||_{L_x^2}||G(s)||_{L_x^2}. && (3.12)\end{aligned}$$

By assumption (3.6) and from the above estimate, we get

$$\begin{aligned} |< U_H(s)^*F(s), U_H(t)^*G(t) >| &\leq ||U_H(t)U_H(s)^*F(s)||_{L_x^\infty}||G(t)||_{L_x^1} && (3.13)\\ &\lesssim \omega(s-t)^\gamma||F(s)||_{L_x^1}||G(s)||_{L_x^1}. && (3.14)\end{aligned}$$

The last equality follows by Holder's inequality and γ is coming from (3.6).

Now by interpolating with

$$|< U_H(s)^*F(s), U_H(t)^*G(t) >| \leq ||F(s)||_{L_x^2}||G(s)||_{L_x^2},$$

we obtain that if r' is defined by $1/r' = 1-\theta+\theta/2,\ 0<\theta<1$, then $1<r'<2$. If we denote by r the dual exponent of r' then $1/r = \theta/2$, and choosing q such that (q,r) is γ-admissible, we will also get that (q',r') is sharp γ-admissible, and that $\gamma(1-\theta) = 2/q$. Therefore,

$$|< U_H(s)^*F(s), U_H(t)^*G(t) >| \leq \omega(t-s)^{\frac{2}{q}}||F(s)||_{L_x^{r'}}||G(s)||_{L_x^{r'}}.$$

Since (q,r) is sharp γ-admissible (and it is not an endpoint), then we can apply weak young inequality, obtaining

$$\begin{aligned} \int\int |< U_H(s)^*F(s), U_H(t)^*G(t) >|\, dsdt &\leq \int\int \omega(t-s)^{\frac{2}{q}}||F(s)||_{L_x^{r'}}||F(s)||_{G_x^{r'}}dsdt\\ &\leq ||\omega(t-s)||_{L_\omega^{\frac{q}{2}}}||F||_{L_t^{q'}L_x^{r'}}||G||_{L_t^{q'}L_x^{r'}},\end{aligned}$$

where $2/q+1/q'+1/q' = 2$, and since ω is in weak L^1, (3.4) follows.

Now to prove (3.5) we can proceed as in [56] (Section 7).

The following result gives us conditions on operator U_H to obtain global in time Strichartz estimates:

Theorem 2. *1. Consider the following restriction on the coefficients*

$$\frac{b_j(t)}{2} - \frac{c_j^2(t)}{4} - \frac{c_j'(t)}{4} = \sigma_j, \qquad j \geqslant 1, \qquad \sigma_j \in \{-\frac{1}{4}, 0, \frac{1}{4}\}. \tag{3.15}$$

The evolution operator U_H associated to the Cauchy problem (1.1)-(1.2) is given by (2.10) where

$$\mu_j(t) = \begin{cases} \frac{\sinh(\omega_j t)}{\omega_j}\ , \ \textit{if } \sigma_j = -1, \\ t\ , \qquad \textit{if } \sigma_j = 0, \\ \frac{\sin(\omega_j t)}{\omega_j}\ , \ \textit{if } \sigma_j = +1, \end{cases} \tag{3.16}$$

and it satisfies

$$\left\| U_H\,(t,s)\,\varphi \right\|_\infty \leq \left(\prod_{j=1}^{d} \frac{1}{2\pi |\mu_j\,(t-s)\,|} \right)^{d/2} \left\| \varphi \right\|_1. \tag{3.17}$$

Furthermore, if $\delta_j = -1$ for some j, we have global in time Strichartz estimates.

Proof. It is easy to see that (3.17) follows from (2.12) and (3.16). The global in time Strichartz estimates follow from Lemma 5 and observing that (as was pointed out in [5] (Section 2)) if $\delta_k = -1$ for some k and $\delta_j = 1$ for $i \neq k$ (the worst of the possible cases), then

$$w(t) = C \left(\frac{1}{|t|} 1_{|t| \leq \delta} + \left(e^{-\omega_k t} \prod_{j \neq k}^{d} \frac{1}{2\pi |\sin(\omega_j t)|} \right)^{\frac{1}{d}} 1_{|t| > \delta} \right) \tag{3.18}$$

is in $L^1_\omega(\mathbb{R})$, and from Lemma 5 we obtain global in time Strichartz estimates. □

4. DAMPED HARMONIC NONLINEAR SCHRÖDINGER EQUATION

All linear exactly solvable models which we discussed in Sections 1 and 2 are of interest in a general treatment of the nonlinear time-dependent Schrödinger equation, see [26], [28], [40], [46]. In this Section we apply the results of the last sections to the study of the nonlinear version of equation (1.1)-(1.2) (with algebraic nonlinearity):

$$i\frac{\partial u}{\partial t} = -\frac{1}{2}\Delta u + \sum_{j=1}^{d} \left(\frac{b_j\,(t)}{2} x_j^2 u - f_j(t) x_j u + i g_j(t) \frac{\partial u}{\partial x_j} - i \frac{c_j\,(t)}{2} \left(2x_j \frac{\partial u}{\partial x_j} - u \right) \right) + h|u|^{p-1}u \tag{4.1}$$

$$u(x,0) = u_0(x). \tag{4.2}$$

It includes the damped harmonic nonlinear Schrödinger equation (1.4) and the following well-known cases:

- The nonlinear Schrödinger equation with zero potential

$$i\frac{\partial u}{\partial t} = -\frac{1}{2} \triangle u + hu|u|^{p-1}. \tag{4.3}$$

- The nonlinear Schrödinger equation with the quadratic potential possibly depending on time [5], [6]
$$i\frac{\partial u}{\partial t}=-\frac{1}{2}\bigtriangleup u+\sum_{j=1}^{d}\frac{b_j\left(t\right)}{2}x_j^2u+hu|u|^{p-1}. \tag{4.4}$$
- The Gross–Pitaevskii equation
$$i\frac{\partial u}{\partial t}=-\frac{1}{2}\triangle u+\frac{m}{2}\omega(t)^2x^2u-f(t)xu+hu|u|^{p-1}, \tag{4.5}$$
see [16], [60], [39].

These equations have different applications, for example, the propagation of waves, optical transmission lines with online modulators, propagation of light beams in nonlinear media with a gradient of the refraction index, or in the theory of Bose-Einstein condensate in trapped gases [16], [60]. The following local well-posedness result in $L_x^2(\mathbb{R}^d)$ in the subcritical sense is a consequence of Strichartz estimates and a boostrap argument [56]:

Proposition 1. *Let p be an L_x^2-subcritical exponent ($0<p-1<4/d$), $h=\pm 1$. Then for any $R>0$ there exists $T>0$ such that for all $u_0\in L_x^2(\mathbb{R}^d)$ in the ball $B_R=\{u_0\in L_x^2(\mathbb{R}^d):\|u_0\|_{L_x^2(\mathbb{R}^d)}<R\}$ there exists a unique strong L_x^2 solution u to (4.1)-(4.2) in the space $S^0([-T,T]\times\mathbb{R}^d)\subset C_t^0L_x^2([-T,T]\times\mathbb{R}^d)$.*

Sketch of the proof: Since we obtained Strichartz estimates, see Section 3, we can follow the proof for the case of the nonlinear Schrödinger equation without potential, see [56], [8] or [54].

Remark 4. *(Time dependent nonlinearity) One can also consider a generalization of the nonautonomous Schrödinger equation*
$$i\frac{\partial u}{\partial t}=-\frac{1}{2}\Delta u+\sum_{j=1}^{d}\left(\frac{b_j\left(t\right)}{2}x_j^2u-f_j(t)x_ju+ig_j(t)\frac{\partial u}{\partial x_j}-i\frac{c_j\left(t\right)}{2}\left(2x_j\frac{\partial u}{\partial x_j}-u\right)\right)+h(t)|u|^{p-1}u \tag{4.6}$$
$$u(x,0)=u_0(x), \tag{4.7}$$
considering for example $h\in C^\infty(\mathbb{R};\mathbb{R})$. This equation will include the nonautonomous Schrödinger equation [8]
$$i\frac{\partial u}{\partial t}=-\frac{1}{2}\triangle u+h(t)u|u|^{p-1}. \tag{4.8}$$

Acknowledgments. The author is grateful to Dr Sergei K. Suslov for his fundamental guidance and support and Dr. Svetlana Roudenko for very useful discussions and key references. The author thanks Dr Luca Fanelli and Dr Don Jones for helpful discussions and Dr. Carlos Castillo-Chávez for his endeavours to open new research opportunities at ASU. Finally, the author is indebted to Professor George A. Hagedorn, for pointing out references relevant to this paper and future research.

References

[1] G.P. Arrighini and N. L. Durante, *More on the quantum propagator of a particle in a linear potential*, Am. J. Phys., 64, 1996, 8, 1036-1041.
[2] H. Bateman, *Partial Differential Equations of Mathematical Physics*, Dover, New York, 1944
[3] L. S. Brown and Y. Zhang, *Path integral for the motion of a particle in a linear potential*, Am. J. Phys., 62, 1994, 9, 806-808.

[4] P. Caldirola, *Forze non conservative nella meccanica quantistica*, Nuovo Cim., 18, 1941, 393-400.
[5] R. Carles, *Global existence results for nonlinear Schrödinger equations with quadratic potentials*, Discrete Contin.Dyn., 13, 2005, 2, 385-398.
[6] R. Carles and Y. Nakamura, *Nonlinear Schrödinger equations with stark potential*, Hokkaido Math. J., 33, 2004, 3, 719-729.
[7] R. Carles, *Nonlinear Schrodinger equation with time dependent potential*, arXiv:0910.4893, [math.AP] 26 Oct 2009.
[8] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, American Mathematical Society, Providence, Rhode Island, 10, 2003.
[9] T. Cazenave and A. Haraux, *An Introduction to semilinear evolution equations*, Oxford Lecture Series in Mathematics and Its Applications, Oxford Science Publications, Claredon Press, Oxford, 13, 1998.
[10] R. Cordero-Soto, M. R. Meiler, and S. K. Suslov, *Solution of the Cauchy problem for a time-dependent Schrödinger equation*, J. Math. Phys. , 49, 2008, 7.
[11] R. Cordero-Soto, R. M. Lopez, E. Suazo and S. K. Suslov, *Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields*, Lett. Math. Phys. , 84, 2008, 2-3, 159-178.
[12] R. Cordero-Soto, E. Suazo and S. K. Suslov, *Models of damped oscillators in quantum mechanics*, Journal of Physical Mathematics, 1 (2009), Article ID S090603, 16 pages.
[13] R. Cordero-Soto, E. Suazo and S. K. Suslov, *Quantum integrals of motion for variable quadratic Hamiltonians*, arXiv:0912.2113v2, [math.AP] 11 Dec 2009.
[14] W. Craig, T. Kappeler, and W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math. 48 (1995), no. 8, 769-860.
[15] H. L Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer Verlag, 1987.
[16] F. Dalfovo, S. Giorgini, L. P. Pitaevskii and S. Stringari, *Hydrodynamic approach to constructing solutions of nonlinear Schrödinger equation in the critical case*, Rev. Mod. Phys. , 71, 1999, 463.
[17] P. Ehrenfest, Bemerkung *Huber die angenaherte Gultigkeit der klassischen Mechanik innerhalb der Quantenmechanik,* Zeitschrift fur Physik A 45 (1927), 455–457.
[18] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals*, McGraw–Hill, New York, 1965, 4.
[19] D. Fujiwara, *A Construction of the fundamental solution for the Schrödinger equation*, J. Anal. Math. , 35, 1979, 41-96.
[20] D. Fujiwara, *On a nature of convergence of some path integrals, I*, Duke Math.J. , 47, 1980, 559-600.
[21] V. N. Gheorghe, F. G. Major, G. Werth, *Charge particle traps*, Springer, 2005.
[22] D. R. Haaheim and F. M. Stein, *Methods of solution of the Riccati differential equation*, Mathematics Magazine, 42, 1969, 2, 233-240.
[23] G. A. Hagedorn, M. Loss, and J. Slawny, *Non-stochasticity of time-dependent quadratic Hamiltonians and spectra of canonical transformations,* J. Phys. A: Math. Gen. 19, 1986, 521-531.
[24] G. A. Hagedorn, *Raising and lowering operators for semiclassical wave package,* Ann. Phys. 269, 1998, 77-104.
[25] B. R. Holstein, *The linear potential propagator*, Am. J. Phys. , 65, 1997, 5, 414-418.
[26] J. Howland, *Scattering theory for Hamiltonians periodic in time*, Indiana Univ. Math. J. , 28, 1979, 3, 471-494.
[27] K. Husimi, *Miscellanea in elementary quantum mechanics*, Progress of Theoretical Physics, 9, 1953, 4, 381-404.
[28] D. R. Jafaev, *Wave operators for the Schrödinger equation*, Teoret. Mat. Fiz. , 45, 1980, 2, 224-234.
[29] E. Kanai, *On the quatization of dissipative systems*, Prog. Theor. Phys. 3, 1941, 440-442.
[30] L. Kapitanski, I. Rodnianski, and K. Yajima *On the fundamental solution of a perturbed harmonic oscillator,* Topol. Methods Nonlinear Anal. 9 (1997), no. 1, 77-106.
[31] M. Keel and T. Tao, *Endpoints Strichartz estimates*, Amer. J. Math. , 120, 1998, 5, 955-980.
[32] C. E. Kenig, G. Ponce, V. Luis, *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations*, Invent. Math., 134 (1998), no. 3, 489–545.
[33] Killip, Rowan; Li, Dong; Visan, Monica; Zhang, Xiaoyi, *Energy-critical NLS with quadratic potential*, Comm. PDE. 34 (2009), 1531-1565.
[34] L. D. Landau and E. M. Lifshitz, *Mechanics*, Pergamon Press, Oxford, 1976.
[35] N. Lanfear and S. K. Suslov, *The time-dependent Schrödinger equation, Riccati equation and Airy functions*, arXiv:0903.3608.

[36] R. M. Lopez and S. K. Suslov, *The Cauchy problem for a forced harmonic oscillator*, arXiv:0707.1902v8 [math-ph], 27 Dec 2007.
[37] E. Merzbacher, *Quantum mechanics*, Third edition, John Wiley Sons, New York, 1998.
[38] W. Miller, *Symmetry and Separation of Variables*, Addison-Wesley, Massachusetts, 1977.
[39] V. M. Perez-Garcia, P. Torres, G. D. Montesinos, *The method of moments for nonlinear Schrödinger equations: theory and applications*, SIAM J. Appl. Math. , 67, 2007, 4, 990-1015.
[40] V. Naibo and A. Stefanov, *On some Schrödinger and wave equations with time dependent potentials*, Math. Ann. , 334, 2006, 2, 325-338.
[41] P. Nardone, *Heisenberg picture in quantum mechanics and linear evolutionary systems*, Am. J. Phys. , 61, 1993, 3, 232-237.
[42] V. S. Popov, A. M. Peremolov, *Parametric Excitation of a Quantum Oscillator I*, SOVIET PHYSICS JETP, 29, 1968, 4, 738-745.
[43] A. M. Perelomov, Y. B. Zel'dovich, *Quantum mechanics: selected topics*, 1999.
[44] M. Reed and B. Simon, *Methods of mathematical physics*, Academic Press, New York, 1975.
[45] R. W. Robinett, *Quantum mechanical time-development operator for the uniformly accelerated particle*, Am. J. Phys. , 64, 1996, 6, 803-808.
[46] I. Rodnianski and W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. , 155, 2004, 3, 451-513.
[47] W. Schlag, *Dispersive estimates for Schrödinger operators: a survey*, arXiv: math/0501037v3 [math.AP], 8, 10 Jan 2005, 4, 409-413, 1058-6458.
[48] A. Stefanov, *Strichartz estimates for the magnetic Schroedinger equation*, Adv. Math. (210) (2007), 246-303.
[49] E. M. Stein and R. Shakarchi, *Real analysis: Measure theory, integration, and Hilbert spaces*, Princeton University Press, Princeton, New Jersey, 2005.
[50] E. Suazo and S. K. Suslov, *An integral form of the nonlinear Schrödinger equation with wariable coefficients*, arXiv:0805.0633v2 [math-ph], 19 May 2008.
[51] E. Suazo, L. Vega and S.K. Suslov, *The Riccati differential equation and a diffusion-type equation*, arXiv:0807.4349, [math-ph], 8 August 2008.
[52] E. Suazo and S. K. Suslov, *Cauchy problem for Schrödinger equation with variable quadratic Hamiltonians*, under preparation.
[53] E. Suazo, *Fundamental solutions of some evolution equations*, Ph.D. Thesis, Arizona State University, 2009.
[54] C. Sulem, P. L. Sulem, *The nonlinear Schrödinger equation: Self-focusing and wave collapse*, Springer-Verlag, New York, 1990.
[55] Ch-I. Um, K-H. Yeon, and T. F. George, *The quantum damped oscillator*, Phys. Rep. 362, 2002, 63-192.
[56] T. Tao, *Nonlinear dispersive equations: Local and global analysis*, CBMS Regional Conference Series in Mathematics, 2006.
[57] Y. Tsutsumi, *L^2-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvak., 30, 1987, 1, 115-125.
[58] K. Yajima, *Smothness and non-smoothness of the fundamental solution of time dependent Schrödinger equations*, Comm. Math. Phys. 181, (1996), no 3, 605-629.
[59] K. Yajima, *Schrödinger evolution equations with magnetic fields*, J Analyse Math. , 56, 1991, 29-76.
[60] Yuri S., Kivshara, Tristram J. Alexander, Sergey K. Turitsyn, *Nonlinear modes of a macroscopic quantum oscillator*, Physics Letters A, 278, 2001, 225-230.
[61] S. Zelditch, *Reconstruction of singularities for solutions of Schrödinger's equation*, Comm. Math. Phys. 90 (1983), no. 1, 1-26.

Department of Mathematics and Statistics, Arizona State University, Tempe, AZ 85287–1804, U.S.A.

E-mail address: esuazo@asu.edu

URL: http://mathpost.asu.edu/~suazo/index.html